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Sortability of Multi-partitions

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Abstract. Recently, the theory of sortability of partition property has been shown to be an effective tool to prove the existence of an optimal partition with that property. In this paper, we extend the theory to multi-partition where the partition is on t types of components. We apply our results to settle an optimal assignment problem whose proof was incomplete as given in the literature.

1. Introduction

Let $N = (\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n)$ denote a set of real numbers, and let $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ denote a partition of N into p nonempty parts π_i. For a given family F of partitions and a given objective function $f(\pi)$, the problem of finding a partition $\pi \in F$ to minimize $f(\pi)$ is known as a *single-partition problem*, or just a *partition problem*, and has been intensively studied in the literature since [1] and [7]. We now extend the study to the case that there are t disjoint lists $N_u = (\theta_{u1} \leqslant \theta_{u2} \leqslant \cdots \leqslant \theta_{un_u})$ for $u = 1, \ldots, t$ and part π_i consists of t types $\pi_i(1), \ldots, \pi_i(t)$ of sub-parts. The problem is then called a *multi-partition problem*, or a *t-partition problem* if t is specified. This problem arises in the study of optimal assembly of components into a system (see Section 4).

F is called a *shape family* if $|\pi_i(u)| = n_{ui}$ is fixed for all i and u, a *size family* if p is fixed and an *open family* if no restriction is imposed, the numbers of partitions in these families are usually exponentially many [8]. A standard approach is to identify some polynomial classes of partition and to prove the existence of an optimal partition in such a class. The method of proving this existence is to show that given an optimal partition not in this class, we can step-by-step change it into a partition in the class while the optimality is preserved in each step. The theory of sortability was developed to assure that step-by-step changes will end at a partition in the class.

Such a class is typically identified by a partition property. For example, a single-type partition is called *consecutive* if for every two parts, $\theta \geqslant \theta'$ for all θ in one part and all θ' in the other part. A multi-partition is *consecutive* if the partition of each type is consecutive. Then the consecutive class consists of all consecutive partitions. The notions of *consistency* and *sortability* are introduced in [10]. A property Q is called *k-consistent* if a necessary and sufficient condition for a partition to satisfy Q is that all subpartitions of k parts satisfy Q. k-sortability is introduced to facilitate a step-by-step change from a partition not satisfying Q to one which does through sorting a set of K parts at each step while such a sorting guarantees the nonincreasingness of the objective function and the decreasingness of a function $s(\pi)$ for some $s(\pi)$. Usually we discuss sortability with respect to a family F.

To avoid dependence of the definition of sortability on the objective function, which varies from problem to problem, we give more specific description on the range of Q-sorting for which $s(\pi)$ is decreased. A *Q-sorting* is a sorting of some given parts to satisfy Q. Q is called *strong-k-sortable* if for every set of k parts not satisfying Q among themselves, every possible Q-sorting of these k parts decreases $s(\pi)$. In this case, the objective function needs to be nonincreasing only for specific set of k parts not satisfying Q, and a specific Q-sorting. There are other levels of k-sortability:

Part-specific. There exists a specific set of k-parts not satisfying Q for which all Q-sorting decreases $s(\pi)$.

Sort-specific. There exists a specific pattern of Q-sorting such that for all k-parts not satisfying Q, such a sorting decreases $s(\pi)$.

Weak. There exists a specific set of k-parts not satisfying Q, and a specific pattern of Q-sorting such that $s(\pi)$ is decreased.

Correspondingly, the objective function needs to be nonincreasing for all sets of k-parts, all choices of Q-sorting, and both for the above three types of sortability, respectively. Let l denote a level. Then the sortability of a property Q has parameters (l, k, F) where F is the underlying family of partitions.

2. Some multi-partition properties

We introduced consecutive multi-partition in the previous section. Note that a consecutive partition of a type induces a linear order of the parts. It is natural and customary to assume that all parts are nonempty. However, in a multi-partition, a nonempty part π_i can have empty sub-parts $\pi_i(u)$ for some u. Thus we need to refine the definition of consecutiveness. Let $\pi_i(u) > \pi_j(u)$ denote the fact that either $\theta \geqslant \theta'$ for all $\theta \in \pi_i(u)$ and all $\theta' \in \pi_j(u)$, or at least one of $\pi_i(u)$ and $\pi_j(u)$ is empty (in the latter case, $\pi_i(u) < \pi_j(u)$ is also correct). We also write $\pi_i > \pi_j$ if $\pi_i(u) > \pi_j(u)$ for all u. Then a multi-partition is *consecutive*(C) if for every pair of parts π_i and π_j, either $\pi_i > \pi_j$ or $\pi_i < \pi_j$. A consecutive multi-partition is called *monotone* (M) if the ordering of all pairs of parts are transitive, namely, there exists a linear ordering of the p parts consistent with the pairwise ordering. A monotone multi-partition is called *index-monotone* (I) if $i > j$ implies $\pi_i > \pi_j$. For example, suppose $N_1 = (\theta_{11} \leqslant \theta_{12} \leqslant \theta_{13} \leqslant \theta_{14})$ and $N_2 = (\theta_{21} \leqslant \theta_{22} \leqslant \theta_{23})$. Then $\pi_1 = (\theta_{11}, \theta_{12}, \theta_{23})$, $\pi_2 = (\theta_{13}, \theta_{14}, \theta_{21}, \theta_{22})$ is consecutive,

$\pi_1 = (\theta_{13}, \theta_{14}, \theta_{23})$, $\pi_2 = (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ is monotone,

$\pi_1 = (\theta_{11}, \theta_{12}, \theta_{21})$, $\pi_2 = (\theta_{13}, \theta_{14}, \theta_{22}, \theta_{23})$ is index-monotone.

Clearly, $I \Rightarrow M \Rightarrow C$.

The indices themselves may be assigned according to some parameter of the problem. For examples, the parameter can be the importance of a part, the cost of a part, or the size of a part. In these cases, we may also use the terms importance-monotone, cost-monotone or size-monotone. But for our problem, the indices are considered fixed.

Let $\sharp_Q(\{n_{u)i}\})$, $\sharp_Q(p,t)$ and $\sharp_Q(t)$ denote the number of shape-t-partitions, size-t-partitions and open-t-partitions, respectively for the class Q.

THEOREM 1.

$$\sharp_I(\{n_{ui}\}) = 1,$$

$$\sharp_I(p,t) = \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} \prod_{u=1}^{t} \binom{n_u + p - 1 - j}{p - 1 - j},$$

$$\sharp_I(t) = \sum_{p=1}^{\max\{n_u\}} \sharp_I(t,p),$$

$\sharp_M \leqslant (p!)\sharp_I$, $\sharp_C \leqslant (p!)^t \sharp_I$ *for any of the three arguments* $\{n_{ui}\}, \{p,t\},$ *or* $\{t\}$.

Proof. $\sharp_I(\{n_{ui}\}) = 1$ is obvious since there is only one way of assigning the smallest n_{u1} elements of type u to part 1, the next smallest n_{u2} elements of type u to part 2, and so on for $u = 1, \ldots, t$. To count $\sharp_I(p,t)$, we note that any consecutive 1-partition can be represented by inserting $p-1$ bars into the elements. Thus type u has $(n_u + p - 1/p - 1)$ varieties by permuting $p-1$ bars with the n_u ordered elements, while the set of elements immediately before the ith bar is $\pi_i(u)$. Since for any part i, necessarily $n_{ui} > 0$ for at least one u, we use inclusion-exclusion formula to discount the cases where a part is empty. Here $p - 1 - j$ is the actual number of bars used to obtain $p - j$ nonempty $\pi_i(u)$.

Note that p must be fixed for I or we wouldn't know which part has what index. Therefore we do not discuss open partition for I. $\sharp_I(t)$ in Theorem 1 merely sums up $\sharp_I(p,t)$ over all p.

$\sharp_M \leqslant (p!)\sharp_I$ since any permutation of the p parts can serve as an index. $\sharp_C \leqslant (p!)^t \sharp_I$ since any permutation of the p parts in a type in an I-partition results in a C-partition. The inequalities are due to the fact that different permutations can contain the same partition when empty $\pi_i(u)$ exist. For example the partition $\pi_1(1) = 1, \pi_2(1) = \pi_1(2) = \emptyset, \pi_2(2) = 2$ is counted in both permutations (1,2) and (2,1). □

Next we study the consistency issue. The smallest k for which Q is k-consistent is called the minimum consistency index of Q. If Q is not k-consistent for all k, we set its minimum consistency index to ∞.

THEOREM 2. *The minimum consistency index is 2 for consecutiveness and index-monotonicity, but ∞ for monotonicity.*

Proof. Suppose π is not consecutive. Then there exist two sub-parts $\pi_i(u), \pi_j(u)$ and elements $a < b < c$, such that $a, c \in \pi_i(u)$ and $b \in \pi_j(u)$. But this implies that π_i and π_j are not consecutive.

Similarly, suppose π is not index-monotone. Then there exist a type u such that either two parts are not consecutive, or $\pi_i(u) > \pi_j(u)$ for some $i < j$. In the former case, the argument for consecutiveness works. In the latter case, π_i and π_j are not index-monotone.

Finally, we show that monotonicity is not k-consistent for all k. Consider $p = k$, and $N_u = \{\theta_{u1} \leqslant \theta_{u2}\}$ for $1 \leqslant u \leqslant t$. Then $\pi_i = \{\theta_{i1}, \theta_{(i+1)2}\}$, $i = 1, \ldots, p-1$, and $\pi_p = \{\theta_{p1}, \theta_{12}\}$ do not satisfy monotonicity, but every $(p-1)\pi_i$ does. For example, for $\pi_1, \ldots, \pi_{p-1}$, the linear order is $\pi_1 > \pi_2 > \cdots > \pi_{p-1}$. □

3. The sortabilities of I and C

We first observe that two results proved in [2] for single-partition also work for multipartition.

LEMMA 3. *Suppose Q is not k-consistent. Then Q is not (l, k, F)-sortable for all l and F.*

LEMMA 4. *Suppose Q' implies Q and Q is k-consistent. Then Q' being (sort-specific, k, F) sortable implies the same for Q.*

From Theorem 2 and Lemma 3, we conclude immediately that M is not (l, k, F) sortable for all l, k, and F. Next we prove that

LEMMA 5. *If $Q \in \{C, I\}$ is not (strong, k, F) sortable, then Q is not so for $k' > k$.*

Proof. It suffices to prove for $k' = k + 1$. Let $F = \{\pi^1, \ldots, \pi^m\}$ be a family of partitions of N not satisfying Q but for every $\pi^i \in F$ not satisfying Q, there exists a set of k parts not satisfying Q and a k-Q-sorting which turns π^i into $\pi^j \in F$. Let N^* be obtained from N by adding $|N|$ new θ_{uj}'s for each u, all greater than θ_{un_u}, and let π^{i*} be obtained from π^i by adding a new part P consisting of these new $\theta_{uj}{'}s$. We consider only sorting in which P remains invariant. Let K denote a k-part not satisfying Q in π^i. Then $K \cup P$ is a $(k+1)$-part not satisfying Q in π^{i*}, and a k-Q-sorting of π^i into π^j corresponds to a $(k+1)$-Q-sorting of π^{i*} to π^{j*}. Hence $F^* = \{\pi^{i*} : \pi \in F\}$ is a family proving Q is not (strong, $k+1$, F) sortable. □

We now prove some results on sortability.

THEOREM 6. *I is strong-k-shape sortable for all $k \geqslant 2$.*

Proof. Define $s(\pi) = \sum_{i=1}^{p} i \sum_{u=1}^{t} \sum_{j \in \pi_{u_i}} j$. Suppose that π contain a set K of k parts not satisfying I. I-sort K to obtain π'. Since π' assigns the smaller j to the part with the smaller index i, it achieves the maximum of $s(\pi)$ on K and is strictly greater than $s(\pi)$ on K since the latter has at least one inversion. Finally, since parts not in K remain unchanged, $s(\pi') < s(\pi)$. □

Clearly, if Q is not (l, k, F) sortable for 1-partition, then it is not so for multi-partition. The following example was given in [2] to demonstrate that consecutiveness is not strongly-k-shape sortable for $k = 3$ (hence all $k \geqslant 3$) in 1-partition: $n = 5$, $p = 4$, $n_1 = 2$, $n_2 = n_3 = n_4 = 1$, $\Pi = \{\pi^1, \pi^2\}$, $\pi_1^1 = \{1, 3\}$, $\pi_2^1 = \{2\}$, $\pi_3^1 = \{4\}$, $\pi_4^1 = \{5\}$, $\pi_1^2 = \{3, 5\}$, $\pi_2^2 = \{2\}$, $\pi_3^2 = \{4\}$, $\pi_4^2 = \{1\}$. π^1 can be sorted into π^2 by C-sorting π_1^1, π_2^2 and π_4^1, while π^2 can be sorted into π^1 by C-sorting $\pi_1^2, \pi_3^2, \pi_4^2$. But neither π^1 nor π^2 is consecutive. By setting $\pi_{ui}^j = \pi_i^j$ for $j = 1, 2$ and $u = 1, \ldots, t$, the above example can be turned into an example that monotonicity is not strongly-k-shape sortable for all $k \geqslant 3$.

Next we show that $Q \in \{C, I\}$ is not (strong, k, size) sortable for all $k \geqslant 2$. By Lemma 5, it suffices to prove for $k = 2$. Let $\Pi = \{\pi^1, \pi^2, \pi^3, \pi^4\}$, where $p = 3$, $t = 4$ and $n_i = 5$ for $i = 1, 2, 3, 4$.

π^1			π^2			π^3			π^4		
13	4	25	123	4	5	12	34	5	1	34	25
1	24	35	13	24	5	123	4	5	123	4	5
13	34	5	1	34	25	13	4	25	123	4	5
123	4	5	12	4	35	1	24	35	13	24	5

No partition in Π satisfies Q. 2-Q-sort parts 1 and 3, then π^1 becomes π^2, π^3 becomes π^4. 2-Q-sort parts 1 and 2, then π^2 becomes π^3, π^4 becomes π^1.

We also show that C is not (part-specific, 2, shape) sortable. Suppose $p = 3$, $t = 4$, each type has 4 elements with $n_{11} = n_{21} = n_{33} = n_{43} = 2$ and all other $n_{uv} = 1$. Let $\Pi = \{\pi^1, \pi^2, \pi^3, \pi^4\}$ where

π^1			π^2			π^3			π^4		
24	3	1	23	4	1	12	4	3	12	3	4
12	4	3	24	1	3	23	1	4	23	4	1
3	4	12	4	3	12	1	3	24	1	4	23
4	1	23	1	4	23	3	4	12	3	1	24

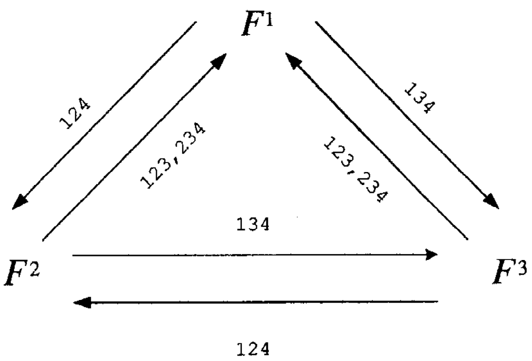
No partition in Π satisfies C. 2-C-sort parts 1 and 2, then π^1 becomes π^2. 2-C-sort parts 1 and 3, then π^2 becomes π^3, π^4 becomes π^1. 2-C-sort parts 2 and 3, then

π^3 becomes π^4. Note that in each case, the pair of parts we sort is the only pair not satisfying Q.

Finally, we show that $Q \in \{I, C\}$ is not (part-specific, k, size) sortable for $k = 3$. Suppose $p = 4$, $t = 3$, each type has 6 elements. Consider these partitions:

π^1				π^2				π^3			
1	23		456	1	23		456	145	23		6
123	45		6	123	45		6	1	45		236
1		23	456	145		23	6	1		23	456
123		45	6	1		45	236	123		45	6
145			236	123			456	123			456

Let F^1 denote a family of 120 partitions obtained by permuting the five types of π^1, and let F^2 and F^3 be obtained from π^2 and π^3 similarly. Define $F = \{F^1, F^2, F^3\}$. Note that no partition in F satisfies Q. For any partition π in F^2, and any three parts of π not satisfying Q, there exists a 3-Q-sort which turns π into π' where π' is in one of the other two F^j family. The labels of the links in the following figure show the set of parts involved in the 3-Q-sorting:



We will use the 12-cell table introduced in [2] to summarize our finding of sortability of a partition property. The 12 cells represent the combinations of four levels and three types.

strong-open	strong-size	strong-shape	sort-sp.-shape	sort-sp.-size	sort-sp.-open
part-sp.-open	part-sp.-size	part-sp.-shape	weak-shape	weak-size	weak-open

In each cell, we give the set $K = \{k : k\text{-sortability is proved.}\}$ and the set $\overline{K} = \{k : k\text{-sortability is disproved.}\}$. It was established in [2] that $k \in K$ in a cell implies the same for all cells below or to the right. The implication of $k \in \overline{K}$ goes the reverse way.

We now give the tables for I, C.

I:

NA	$\overline{K} = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	NA
NA	$\overline{K} = \{3\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	NA

NA: not applicable.

C:

$\overline{K} = \{k \geqslant 2\}$	$\overline{K} = \{k \geqslant 2\}$	$\overline{K} = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$
$\overline{K} = \{2, 3\}$	$\overline{K} = \{2, 3\}$	$\overline{K} = \{k = 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$	$K = \{k \geqslant 2\}$

4. Special cases

We can prove more on sortability if some additional conditions are imposed.

THEOREM 7. *For $t = 2$, M is (part-specific, 2, size) sortable.*

Proof. We prove Theorem 7 by induction on p. Theorem 7 is trivially true for $p = 1, 2$. We now prove for general $p \geqslant 3$.

C-sort $\pi_1(1), \dots, \pi_p(1)$ into $\pi_{[1]}(1) < \cdots < \pi_{[p]}(1)$. Let e denote the index of the element in $\pi_{[1]}(2)$ such that $\theta_{21}, \theta_{22}, \dots, \theta_{2e}$ are in $\pi_{[1]}(2)$, but not $\theta_{2,e+1}$. If θ_{21} is not in $\pi_{[1]}(2)$, set $e = 0$. Set $e = \infty$ if $\pi_{[1]}(2) = \{\theta_{21}, \theta_{22}, \dots, \theta_{2e}\}$. Note that e is defined only when $\pi_{[1]}(1)$ is consecutive, meaning $\pi_{[1]}(1) = \{\theta_{1j} : 1 \leqslant j \leqslant w$, for some $w\}$.

Define $s(\pi) = e$. We prove that if $s(\pi) < \infty$, then every M-2-sorting increases $s(\pi)$; or at least it will after a sequence of steps. Since e can take at most $n - 2$ values, eventually $e = \infty$, or equivalently, both $\pi_{[1]}(1)$ and $\pi_{[1]}(2)$ are consecutive with the smallest elements in types 1 and 2, respectively. Use induction to obtain an M-partition on $\bigcup_{i=2}^{p} \pi_{[i]}$. The partition plus $\pi_{[1]}$ then yields an M-partition for the p parts.

Let $\pi_{[1]}$ be the part containing $\theta_{2,e+1}$. M-sort $(\pi_{[1]}, \pi_{[i]})$ to obtain π'. Consider two cases:

(i) $\pi'_{[1]}(1)$ is consecutive (this must happen if $i = 2$). Then $\pi_{[1]}(2)$ is either consecutive or it contains $\theta_{21}, \theta_{22}, \dots, \theta_{2,e+1}, \theta_{2,x}$ for some $x > e+2$. Therefore $s(\pi') \geqslant e + 1 > s(\pi)$.

(ii) $\pi_{[1]}(1)$ is not consecutive by picking up some smallest elements from $\pi_{[i]}(1)$ (hence $i \geqslant 3$).

Suppose that elements $\theta_{21}, \theta_{22}, \dots, \theta_{2e'}$ are in $\pi_{[1]}(2)$ but not $\theta_{2,e'+1}$ (if $\pi_{[1]}(2) = \emptyset$, then $e' = 0$). Use induction to obtain an M-partition on $\bigcup_{j<i} \pi'_{|j|}$. Let π^2 denote this partition plus $\pi'_{[i]}$. Then $\pi^2_{[1]}(1)$ is consecutive. $\pi^2_{[1]}(2)$ is either consecutive or it contains $\theta_{21}, \theta_{22}, \dots, \theta_{2,e'}, \theta_{2x}$ for some $x > e' + 1$. In the former case $s(\pi) = \infty > s(\pi)$. In the latter case, I-sort $(\pi^2_{[1]}, \pi^2_i)$ to obtain π^3. If $\pi^3_{[1]}(1)$ is consecutive, then as before, either $\pi^3_{[1]}(2)$ is consecutive or $s(\pi^3) = \max\{e+1, e'+1\}$. In either case $s(\pi^3) > s(\pi)$. If $\pi^3_{[1]}(1)$ is not consecutive by picking up some smallest elements from $\pi^3_{[i]}(1)$. Then we iterate as before by using induction on $\bigcup_{j<i} \pi^3$. Note that in each such iteration, the number of elements in $\pi_{[i]}(1)$ decrease. Since that number has a lower bound 0, eventually the iteration must stop, meaning for some r, $\pi^r_{[1]}(1)$ is consecutive and $s(\pi^r) > s(\pi)$. □

COROLLARY 8. *For $t = 2$, I is (part-specific, 2, size) sortable.*

Proof. By replacing $\pi_{[i]}$ with π_i in the proof of Theorem 7. □

A type k will be called a universal type if $|\pi_i(k)| > 0$ for all i. Let Q^u denote Q conditional on the existence of a universal type. Let Q^a denote Q conditional on every type is universal.

THEOREM 9. *The minimum consistency index is 2 for M^u and M^a.*

Proof. Suppose every pair of parts are monotone. Let type k be a universal type. Then every pair of parts is ordered in type k. The ordering of pairs in other types must follow this ordering or that pair would not be monotone. Since this pairwise ordering is transitive, it implies a linear ordering on the parts. □

THEOREM 10. *I^a is (part-specific, k, size) sortable for all $k \geqslant 2$.*

Proof. It is easily verified that if there exist k parts in π not satisfying I^a, then there exist k consecutive parts K not satisfying I^a. I-sort K to obtain π'. Define $s(\pi)$ to be the number of inversions in π, i.e., an inversion occurs if $x \in \pi_i$, $y \in \pi_j$ and $(x-y)(i-j) < 0$. Then the number of inversions is intact in $\pi \backslash K$ but decreases in K. Hence $s(\pi') < s(\pi)$. □

THEOREM 11. *M^u (hence M^a) is (sort-specific, k, shape) sortable for all $k \geqslant 2$.*

Proof. Follows from Lemma 4 since I is (sort-specific, k, shape) sortable for all $k \geqslant 2$ and M^u is 2-consistent. □

5. An application to optimal assembly

Consider a coherent system consisting of p *series modules*, meaning a module works if and only if all components work. There are t types of components and module i needs n_{ui} components of type u for $u = 1, \ldots, t$. Components of the same type are functionally interchangeable. Define $n_u = \sum_{i=1}^{p} n_{ui}$ and $n = \sum_{u=1}^{t} n_u$. Let the n_u components of type u have respective reliabilities $\theta_{u1} \leqslant \theta_{u2} \leqslant \cdots \leqslant \theta_{un_u}$. The problem is to assemble the components into the p modules to maximize the system reliability. Clearly, this is a t-partition problem.

Since the first paper by Derman et al. [3], there has been a large body of literature [4, 5, 6, 11] on this optimal assembly problem, proving that monotone assembly, or its simplified versions, is optimal under various conditions. This line of research culminated in Hwang and Rothblum [9] who gave the weakest sufficient conditions requiring only that the system be *coherent* (essentially meaning no module is without impact on system reliability). They proved the optimality of monotone partition by first proving that if π is a non-monotone optimal 2-partition, then any M-2-sorting preserves optimality. The sortability approach is implicit in the extension from 2 to p. Since M is not 2-sortable, even weakly, a fictitious universal type is introduced to turn M to M^u. Set $s(\pi) = \sum_{i=1}^{p} \sum_{u=1}^{t} \max \pi_i(u)$, where $\max \pi_i(u) = \max\{\theta_{uj} \in \pi_i(u)\}$. They showed that by properly choosing the parts, $s(\pi)$ decreases during every M^u-2-sorting. But this is not so in two cases:

Case (i). Suppose $t = 1$ and n_1 or $n_2 = 1$. For example, $\pi = \{\theta_1, \theta_3\}$ and $\pi_2 = \{\theta_2\}$. The M^u (or consecutive)-sorting yields $\pi_1' = \{\theta_1, \theta_2\}$ and $\pi_2' = \{\theta_3\}$. But $\{\max \pi_1, \max \pi_2\} = \{\theta_2, \theta_3\}$ before and after sorting. This example can be extended to $t > 1$ by having other types satisfying M^u and staying put during sorting.

Case (ii). Suppose π_i and π_j are consecutive but not monotone. Then the M^u-sorting does not decrease $s(\pi)$. For example, $\pi_1 = (\{\theta_{11}, \theta_{12}\}, \{\theta_{23}, \theta_{24}\})$, $\pi_2 = (\{\theta_{13}, \theta_{14}\}, \{\theta_{21}, \theta_{22}\})$. After the monotone-sorting $\pi_1' = (\{\theta_{11}, \theta_{12}\}, \{\theta_{21}, \theta_{22}\})$ $\pi_2' = (\{\theta_{13}, \theta_{14}\}, \{\theta_{23}, \theta_{24}\})$. Again $s(\pi)$ remains unchanged. Note that the problem in Case (i) can be resolved by changing the statistics $\max \pi_i(u)$ to range $\pi_i(u)$. But the problem in Case (ii) is not affected by this change.

By showing M^u is (sort-specific, 2, shape) sortable (Theorem 11), we finally justify the extension from two parts to p parts.

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References

1. Chakravarty, A.K., Orlin, J.B. and Rothlbum, U.G. (1982), A partitioning problem with additive objective with an application to optimal inventory grouping for joint replenishment, *Oper. Res.* 30: 1018–1022.
2. Chang G.J. et al. (1999), Sortabilities of partition properties, *J. Combin. Opt.* 2: 413–427.
3. Derman, C., Lieberman, G.J. and Ross, S.M. (1972), On optimal assembly of systems, *Naval Res. Logist. Quart*. 19: 564–574.
4. Du, D.Z., When is a monotonic grouping optimal? in: Osalei, S. and Cas, J. (eds), *Reliability Theory and Applications*, World Scientific, New Jersey, pp. 66–76.
5. Du, D.Z. and Hwang, F.K. (1990), Optimal assembly of an s-stage k-out-of-n system, *SIAM J. Disc. Math.* 3: 349–354.
6. Hollander, M., Proschan, F. and Sethuraman, J. (1977), Functions decreasing in transportation and their applications in ranking problems, *Ann. Statist.* 5: 722–733.

7. Hwang, F.K. (1981), Optimal partitions, *J. Opt. Thy. and Appl.* 34: 1–10.
8. Hwang, F.K. and Mallows, C.L. (1995), Enumerating consecutive and nested partitions, *J. Combin. Thys., Series A* 70: 1–23.
9. Hwang, F.K. and Rothblum, U.G. (1994), Optimality of monotone assemblies for coherent systems composed of series modules, *Oper. Res.* 42: 709–720.
10. Hwang, F.K., Rothblum, U.G. and Yao, Y.C. (1996), Localizing combinatorial properties of partitions, *Disc. Math.* 160: 1–23.
11. El-Neweihi, E., Proschan, F. and Setheraman, J. (1987), Optimal assembly of systems using Schur functions and majorization, *Naval Res. logist. Quart.* 34: 705–712.